8 Derivative functions

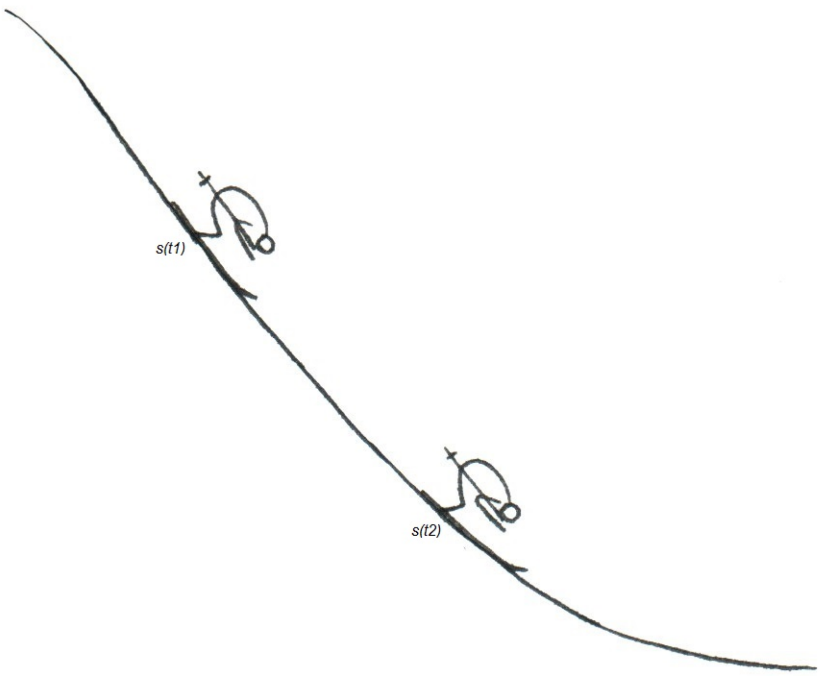
Objectives :
- Define the derivative function
- Know the derivative functions of common functions
- Be able to use the formulas for the derivative of a sum, a product, and a quotient
- Study the derivatives of more complex functions using these formulas

1. Derivative Function

Let f be a function defined and differentiable at every a in an interval I.
The hypothesis "differentiable" means that for $a \in I$, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.
We called this limit the « derivative number of f at a » in Chapter **??**, and we denoted it by $f'(a)$.

> **Définition 8.1** Let f be a function differentiable at every x in an interval I, then the function that associates x with $f'(x)$ is called the *derivative function* of f on I. It is denoted by f'.

Example : If we consider the "position" function, denoted by $t \mapsto s(t)$ (in this case, we consider a "curvilinear abscissa"), which associates a solid's position on its trajectory to an instant t, then :
- The derivative function of $s(t)$ corresponds to the velocity of the solid, so we denote it by $v(t)$ instead of $s'(t)$.
- We can even look for the derivative function of $v(t)$, which corresponds to the acceleration of the solid, and we denote it by $a(t)$ rather than $v'(t)$ or $s''(t)$.



2. Derivatives of Common Functions

A. Constant Function

Let $k \in \mathbb{R}$ and $f : x \mapsto k$, for $x \in \mathbb{R}$. For $h \neq 0$, $\frac{f(x+h)-f(x)}{h} = \frac{k-k}{h} = 0$.
Thus, f is differentiable on $\mathbb{R}$, and for all $x \in \mathbb{R}$, $f'(x) = 0$.

Propriété 8.1 The derivative of a constant function is the zero function.

B. The Function $x \mapsto x^n, n \in \mathbb{N}^*$

Propriété 8.2 Let $n \in \mathbb{N}^*$ and f be the function defined on $\mathbb{R}$ by $f(x) = x^n$.
Then f is differentiable on $\mathbb{R}$, and for $x \in \mathbb{R}$, we have $f'(x) = nx^{n-1}$.

Démonstration *Idea of the proof :*
Let $x \in \mathbb{R}$ and h be a nonzero real number. Let's compute the quotient $\frac{f(x+h)-f(x)}{h}$. First, we have $f(x+h) = (x+h)^n$. Expanding this expression, we obtain terms in x^n, $x^{n-1} \times h$, $x^{n-2} \times h^2$, ..., and h^n. By carefully analyzing how the product $(x+h)(x+h)\dots(x+h)$ expands, we notice that the term x^n appears only once, and the term $x^{n-1} \times h$ appears n times. Thus, we have :

$$(x+h)^n = x^n + n \times x^{n-1}h + \cdots + x^{n-2}h^2 + \cdots + h^n$$

So :

$$\frac{f(x+h)-f(x)}{h} = \frac{n \times x^{n-1}h + \cdots + x^{n-2}h^2 + \cdots + h^n}{h} = nx^{n-1} + hQ(x,h)$$

where $Q(x,h)$ is a polynomial expression depending on x and h, whose limit as h approaches 0 exists. Thus, we obtain :

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \left(nx^{n-1} + hQ(x,h)\right) = nx^{n-1}$$

Remark : We therefore obtain the following results :
- If $f(x) = x$, then for $x \in \mathbb{R}$, we have $f'(x) = 1$;
- If $f(x) = x^2$, then for $x \in \mathbb{R}$, we have $f'(x) = 2x$;
- If $f(x) = x^3$, then for $x \in \mathbb{R}$, we have $f'(x) = 3x^2$;
- ...

Example : Let f be the function defined on $\mathbb{R}$ by $f(x) = x^3$. Determine the equation of the tangent to $\mathscr{C}_f$ at the point with abscissa 1.
The slope of the tangent to $\mathscr{C}_f$ at the point with abscissa 1 is given by $f'(1)$. To calculate $f'(1)$, we can use two methods :
- The definition of the derivative : it is the limit as h approaches 0 of the quotient $\frac{f(1+h)-f(1)}{h}$;
- Or, more quickly, the derivative function of f : for all $x \in \mathbb{R}$, we have $f'(x) = 3x^2$, so $f'(1) = 3 \times 1^2 = 3$.

Moreover, we have $f(1) = 1^3 = 1$. The equation of T_1 is therefore : $y = f'(1)(x-1) + f(1)$, that is, $y = 3(x-1) + 1$, or, simplifying : $T_1 : y = 3x - 2$.

C. Reciprocal Function

Propriété 8.3 Let f be the function defined on $\mathbb{R}^*$ by $f(x) = \frac{1}{x}$.
Then f is differentiable on $\mathbb{R}^*$, and for $x \neq 0$, we have $f'(x) = -\frac{1}{x^2}$.

Démonstration For $x \neq 0$ and $h \neq 0$ such that $x + h \neq 0$, we have :

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \frac{-h}{x(x+h)h} = -\frac{1}{x(x+h)}$$

Thus, we obtain :

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\left(-\frac{1}{x(x+h)}\right)=-\frac{1}{x^2}$$

Example : Let f be the reciprocal function : for $x\neq 0$, $f(x)=\frac{1}{x}$. Determine an equation of the tangent to $\mathscr{C}_f$ at the point with abscissa 1.
This tangent T_1 has the equation $y=f'(1)(x-1)+f(1)$.
To determine it, we need $f'(1)$ and $f(1)=\frac{1}{1}=1$.
For all $x\neq 0$, we have $f'(x)=-\frac{1}{x^2}$, so $f'(1)=-\frac{1}{1^2}=-1$.
Thus, T_1 has the equation $y=-1\times(x-1)+1$, or equivalently, $T_1 : y=-x+2$.

D. Square Root Function

Propriété 8.4 Let f be the function defined on $\mathbb{R}_+$ by $f(x)=\sqrt{x}$.
Then f is differentiable on $\mathbb{R}_+^*$, and for $x\in\mathbb{R}_+^*$, we have $f'(x)=\frac{1}{2\sqrt{x}}$.
Caution : f is not differentiable at 0.

Démonstration For $x\in\mathbb{R}_+^*$ and $h\in\mathbb{R}_+^*$, we have :

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{\sqrt{x+h}^2-\sqrt{x}^2}{h(\sqrt{x+h}+\sqrt{x})}=\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

Thus, we obtain :

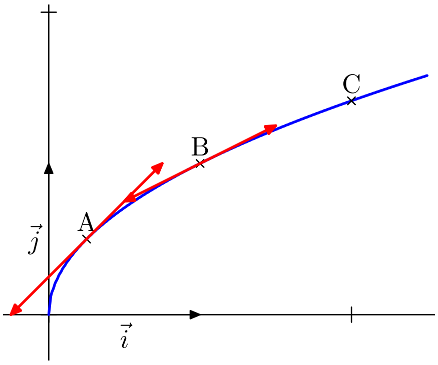
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

If $x=0$, for $h>0$, we have $\frac{f(0+h)-f(0)}{h}=\frac{\sqrt{h}}{h}=\frac{1}{\sqrt{h}}$. As h tends to 0, $\sqrt{h}$ also tends to 0, so $\frac{1}{\sqrt{h}}$ takes increasingly large values. Therefore, the limit as h tends to 0 of the quotient $\frac{f(0+h)-f(0)}{h}$ does not exist : the square root function is not differentiable at 0.

Application to Curve Plotting :
To plot the curve representing the square root function, we create a table of values. For each point in this table, we calculate the slope of the tangent to the curve at that point and then determine the equation of the tangent :

a	$\frac{1}{4}$	1	2
$f(a)$	$\frac{1}{2}$	1	$\sqrt{2}$
$f'(a)$	1	$\frac{1}{2}$	$\frac{1}{2\sqrt{2}}$

- At $a=\frac{1}{4}$, the equation of the tangent is : $y=1\times(x-\frac{1}{4})+\frac{1}{2}$, which simplifies to $y=x+\frac{1}{4}$;
- At $a=1$, the equation of the tangent is : $y=\frac{1}{2}(x-1)+1$, which simplifies to $y=\frac{1}{2}x+\frac{1}{2}$;
- At $a=2$, the equation of the tangent is : $y=\frac{1}{2\sqrt{2}}(x-2)+\sqrt{2}$, which simplifies to $y=\frac{x\sqrt{2}}{4}+\frac{\sqrt{2}}{2}$.



3. Operations on Differentiable Functions

A. Derivative of a Sum

Propriété 8.5 Let u and v be two differentiable functions on an interval I.
Let f be the function defined on I by $f(x) = u(x) + v(x)$ (we also write $f = u + v$ on I).
Then the function f is differentiable on I, and for $x \in I$, $f'(x) = u'(x) + v'(x)$. We write $f' = u' + v'$.

Example : Let f be the function defined on $\mathbb{R}$ by $f(x) = x^3 + x^2 + 3$.
f is differentiable on $\mathbb{R}$ as the sum of differentiable functions on $\mathbb{R}$, and for $x \in \mathbb{R}$, we have : $f'(x) = 3x^2 + 2x$.

B. Product by a Real Number

Propriété 8.6 Let u be a differentiable function on an interval I, and λ a real number.
Let f be the function defined on I by $f(x) = \lambda u(x)$ (we write $f = \lambda u$ on I).
Then the function f is differentiable on I, and for $x \in I$, $f'(x) = \lambda u'(x)$. We write $f' = \lambda u'$.

Example : Let f be defined on $\mathbb{R}$ by $f(x) = 2x^2$, and g defined on $\mathbb{R}$ by $g(x) = 4x^3 - 2x$.
Then, $f'(x) = 2 \times 2x$ and $g'(x) = 4 \times 3x^2 - 2$.

Consequence :
Polynomial functions are therefore differentiable on their domain.

C. Derivative of a Product

Propriété 8.7 Let u and v be two differentiable functions on an interval I.
Let f be the function defined on I by $f(x) = u(x)v(x)$.
Then, f is differentiable on I, and for $x \in I$, $f'(x) = u'(x)v(x) + u(x)v'(x)$. We write $f' = u'v + uv'$.

Démonstration (*Idea of the proof*) For $a \in I$ and $h \neq 0$ such that $a + h \in I$, we have :

$$\begin{aligned}
\frac{(uv)(a+h) - (uv)(a)}{h} &= \frac{u(a+h).v(a+h) - u(a).v(a)}{h} \\
&= \frac{u(a+h).v(a+h) - u(a).v(a+h) + u(a).v(a+h) - u(a).v(a)}{h} \\
&= v(a+h) \times \frac{u(a+h) - u(a)}{h} + u(a) \times \frac{v(a+h) - v(a)}{h}
\end{aligned}$$

Now, we have $\lim\limits_{h \to 0} v(a+h) = v(a)$,
$\lim\limits_{h \to 0} \frac{u(a+h)-u(a)}{h} = u'(a)$,
and $\lim\limits_{h \to 0} \frac{v(a+h)-v(a)}{h} = v'(a)$.
Assuming (under certain conditions satisfied here) that the limit of a product is the product of the limits (and similarly for the sum), we get :

$$\lim_{h \to 0} \frac{(uv)(a+h) - (uv)(a)}{h} = v(a) \times u'(a) + u(a) \times v'(a)$$

This is true for all $a \in I$, so on I we have $f' = u'v + uv'$.

Example : Let f be the function defined on $\mathbb{R}_+$ by $f(x) = x^3\sqrt{x}$.
f is differentiable on $\mathbb{R}_+^*$ as the product of differentiable functions : f can be written as $u \times v$ with
$\begin{cases} u(x) = x^3 \\ v(x) = \sqrt{x} \end{cases}$, where u is differentiable on $\mathbb{R}$ and v is differentiable on $\mathbb{R}_+^*$.

We have therefore : $\begin{cases} u'(x) = 3x^2 \\ v'(x) = \frac{1}{2\sqrt{x}} \end{cases}$.
With these notations, we have $f' = u'v + uv'$ so :

$$\text{For } x > 0,\ f'(x) = u'(x)v(x) + u(x)v'(x) = (3x^2)\sqrt{x} + x^3 \times \frac{1}{2\sqrt{x}} = 3x^2\sqrt{x} + \frac{x^3}{2\sqrt{x}}.$$

Simplifying, we get :

$$f'(x) = 3x^2\sqrt{x} + \frac{1}{2}x^3 \times \frac{\sqrt{x}}{\sqrt{x}\sqrt{x}} = x^2\sqrt{x} + \frac{1}{2}x^2\sqrt{x} = \frac{7}{2}x^2\sqrt{x}$$

Remark : In the case of this example, property 7.7 allows us to state that f is differentiable on $\mathbb{R}_+^*$, but it does not allow us to conclude about the differentiability of f at 0.
To do so, we return to the definition of the derivative : f is differentiable at 0 if the quotient $\frac{f(0+h)-f(0)}{h}$ has a real limit as $h \to 0$. We have :

$$\frac{f(0+h)-f(0)}{h} = \frac{h^3\sqrt{h}}{h} = h^2\sqrt{h}$$

Therefore :

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} h^2\sqrt{h} = 0$$

Thus, f is differentiable at 0 and $f'(0) = 0$.

Propriété 8.8 (Consequence) Let u be a differentiable function on an interval I.
Let f be the function defined on I by $f(x) = (u(x))^2$.
Then the function f is differentiable on I and for all $x \in I$, we have $f'(x) = 2 \times u(x) \times u'(x)$. We write :

$$\left(u^2\right)' = 2uu'$$

D. Derivative of a quotient

Propriété 8.9 Let u and v be two differentiable functions on an interval I, with $v(x) \neq 0$ for $x \in I$.
Let f be the function defined on I by $f(x) = \frac{u(x)}{v(x)}$.
Then, f is differentiable on I and for $x \in I$, $f'(x) = \frac{u'(x)v(x)-u(x)v'(x)}{(v(x))^2}$. We write $f' = \frac{u'v-uv'}{v^2}$.

Démonstration (*Idea of the proof*) For $a \in I$ and $h \neq 0$ such that $a+h \in I$, we have :

$$\begin{aligned}
f(a+h) - f(a) &= \frac{u(a+h)}{v(a+h)} - \frac{u(a)}{v(a)} \\
&= \frac{u(a+h).v(a) - u(a).v(a+h)}{v(a+h).v(a)} \\
&= \frac{u(a+h).v(a) - u(a).v(a) + u(a).v(a) - u(a).v(a+h)}{v(a+h).v(a)} \\
&= \frac{1}{v(a+h).v(a)} \times (v(a) \times (u(a+h) - u(a)) + u(a) \times (v(a) - v(a+h)))
\end{aligned}$$

We divide the entire expression by h, and we get :

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{1}{v(a+h).v(a)} \times \frac{v(a) \times (u(a+h) - u(a)) + u(a) \times (v(a) - v(a+h))}{h} \\
&= \frac{1}{v(a+h).v(a)} \times \left(v(a) \times \frac{u(a+h)-u(a)}{h} + u(a) \times \frac{v(a)-v(a+h)}{h}\right)
\end{aligned}$$

Now we have $\lim\limits_{h\to 0} v(a+h) = v(a)$,
$\lim\limits_{h\to 0} \frac{u(a+h)-u(a)}{h} = u'(a)$,

and $\lim\limits_{h\to 0} \frac{v(a+h)-v(a)}{h} = v'(a)$.
Assuming (under certain conditions satisfied here) that the limit of a product is the product of the limits (and similarly for the sum and quotient), we get :

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \frac{v(a)\times u'(a) - u(a)\times v'(a)}{(v(a))^2}$$

This is true for all $a \in I$, so on I we have $f' = \frac{u'v-uv'}{v^2}$.

Example : Let f be the function defined on $\mathbb{R}$ by $f(x) = \frac{3x-4}{x^2+3}$.
f is differentiable on $\mathbb{R}$ as a quotient of differentiable functions on $\mathbb{R}$ whose denominator does not vanish : we have $f = \frac{u}{v}$ with

$$\begin{cases} u(x) = 3x-4 \\ v(x) = x^2+3 \end{cases}$$

We have therefore :

$$\begin{cases} u'(x) = 3 \\ v'(x) = 2x \end{cases}$$

Thus, $f' = \frac{u'v-uv'}{v^2}$. So :

$$\text{For } x\in\mathbb{R},\ f'(x) = \frac{3\times(x^2+3)-(3x-4)\times(2x)}{x^2+3^2} = \frac{-3x^2+8x+9}{x^2+3^2}$$

Consequence :
Rational functions (quotients of two polynomials) are differentiable on their domain.

Propriété 8.10 (Consequence) Let u be a function that is defined, differentiable, and nonzero on an interval I.
Let f be the function defined on I by $f(x) = \frac{1}{u(x)}$.
Then f is differentiable on I and for $x \in I$, we have $f'(x) = -\frac{u'(x)}{u(x)^2}$. We write :

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

4. Formula Sheet

In the following formula sheet, k is any fixed real number and n is a nonzero natural integer.

Function f	Derivative f'	Domain of differentiability of f
$x \mapsto k$	$x \mapsto 0$	$\mathbb{R}$
$x \mapsto x$	$x \mapsto 1$	$\mathbb{R}$
$x \mapsto x^2$	$x \mapsto 2x$	$\mathbb{R}$
$x \mapsto x^n$	$x \mapsto nx^{n-1}$	$\mathbb{R}$
$x \mapsto \sqrt{x}$	$x \mapsto \dfrac{1}{2\sqrt{x}}$	$\mathbb{R}_+^*$
$x \mapsto \dfrac{1}{x}$	$x \mapsto -\dfrac{1}{x^2}$	$\mathbb{R}^*$
$x \mapsto \cos(x)$	$x \mapsto -\sin(x)$	$\mathbb{R}$
$x \mapsto \sin(x)$	$x \mapsto \cos(x)$	$\mathbb{R}$

Some reminders :
- If u and v are differentiable on I, then $u+v$ is differentiable on I and $(u+v)' = u'+v'$;
- If f is differentiable on I, then kf is differentiable on I, and $(kf)' = kf'$;
- If u and v are differentiable on I, then uv is differentiable on I and $(uv)' = u'v+uv'$;
- If u and v are differentiable on I, with $v(x) \neq 0$ for $x \in I$, then $\frac{u}{v}$ is differentiable on I and $(\frac{u}{v})' = \frac{u'v-uv'}{v^2}$.